Fixed Points and Social Equilibrium Existence without Convexity Conditions: Including an Application for the Default Economy

Ken URAI*

and

Akihiko YOSHIMACHI

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Abstract

The purpose of this paper is to show the existence of equilibrium for economic models with non-convex constraint correspondences. Our proof is based on a new fixed point theorem (a generalization of Kakutani-Fan-Glicksberg) depending merely on conditions for local directions of correspondences, so that we may obtain a general condition which may not depend on any global continuity and convexity on values of mappings for the existence of economic equilibria. We also apply the result for the existence of competitive equilibrium in economies with default and/or bankruptcy.

Keywords : Fixed point theorem, Abstract economy, Non-convexity, Competitive equilibrium, Default, Bankruptcy.
JEL classification: C62; D51; D52; D82; G33

*Graduate School of Economics, Osaka University, Machikaneyama, Toyonaka, Osaka 560-0043, JAPAN. (Internet e-mail address: urai@econ.osaka-u.ac.jp)

1 INTRODUCTION

In this paper, we prove the existence of equilibrium for an abstract economy with non-convex constraint correspondences. Moreover, we apply the theorem to the existence of general equilibrium problem in economies with default and/or bankruptcy.

The method we have used here is based on a new fixed point theorem depending merely on the concept of local directions of correspondences (Section 2, Theorem 1), so that we may obtain a general condition which does not depend on any global continuity and/or convexity on values of mappings for the existence of social equilibria (Section 2, Theorem 2). We further generalize the result (Section 2, Corollary 2.1), and apply it to the general equilibrim existence problem for the default economy (Section 3, Theorem 3) which, as a generalized model for economies with asymmetric information (see, e.g., Dubey-Geanakoplos-Shubik (2000)), is one of the most important framework under the recent development in the general equilibrium theory.

Under the settings of social equilibrium, our result may be stated as follows: If we select appropriately for each player i and a strategy profile $z = (z_j)_{j\in I}$, a non-empty subset $M_i(z)$ of (possibly) non-convex constraint set $K_i(z)$, the condition for existence of equilibra may be obtained merely as a local condition on directions to the set $M_i(z)$ and the better set $P_i(z)$ from z_i for each i which is weaker than to assume any convexity and/or global continuity. Hence, in the competitive equilibrium framework (Section 3, Theorem 3), if we identify $M_i(z)$ with the budget constraint without default and $K_i(z)$ with the constraint allowing for default, the equilibrium existence condition may directly be characterized as consumer i's tastes for keeping promises (moral actions and plans) under exogenously specified default penalties.

It seems somewhat incredible that we have shown the existence of equilibrium without using any convexity condition for budgets. As we shall see in Section 2, mathematically, it is not the convexity but the direction of correspondences that is important for the existence of fixed points, so that, the convexity for the budget is superfluous as long as we use conditions for the direction of better sets and constraint sets.

2 EQUILIBRIUM FOR AN ABSTRACT ECONOMY

In this section, we show the existence of equilibrium for an abstract economy having non-convex constraints. Subsection 2.1 is devoted to develop a fixed point theorem based on the concept of local directions of correspondences. Theorem 2 in subsection 2.2 provides a general result on economic equilibrium existence in which the choice space for the abstract economy is taken to be a subset of locally convex Hausdorff topological vector space E having topological dual E'. Constraint correspondences are taken to be (possibly) non-convex. Preferences of agents are also taken to be (possibly) non-ordered and non-convex.

They are not even supposed to be continuous in the ordinary sense. What we have assumed here is merely conditions on local directions of correspondences.[1]

2.1 A Fixed Point Theorem

Let E be a locally convex Hausdorff topological vector space having topological dual E'. We say that a correspondence $\varphi : E \to E$ has a *fixed direction* $p^x \in E'$ *near* $x \in E$ if there exists a neighbourhood $U(x)$ of x such that for all $z \in U(x)$ and $y \in \varphi(z)$, $\langle p^x, y - z \rangle > 0$. We also say that a correspondence $\varphi : E \to E$ has a *continuous direction near* x if there exists a neighbourhood $U(x)$ of x and a continuous function $p^x : U(x) \to E'$ such that $\langle p^x(z), y - z \rangle > 0$ for all $z \in U(x)$ and $y \in \varphi(z)$, where E' is given a topology of compact convergence. Moreover, we say that a correspondence $\varphi : E \to E$ has a *compact valued upper semi-continuous direction near* x if there exists a neighbourhood $U(x)$ of x and a non-empty compact valued upper semi-continuous correspondence $p^x : U(x) \to E'$ such that $\langle q, y - z \rangle > 0$ for all $z \in U(x)$, $q \in p^x(z)$ and $y \in \varphi(z)$, where the topology on E' is a topology of compact convergence.

It is clear that if φ has a fixed direction near x, then φ has a continuous direction near x, and if φ has a continuous direction near x, then φ has a compact valued upper semi-continuous direction near x. The first result of this paper is the following fixed point theorem which, as we shall see below, may be considered as a generalization of Kakutani-Fan-Glicksberg's fixed point theorem.

THEOREM 1. *Let X be a non-empty compact convex subset of a locally convex Hausdorff topological vector space E having topological dual, E'. Let $\varphi : X \to X$ be a non-empty valued correspondence. Suppose that φ has a compact valued upper semi-continuous direction near x for each x such that $x \notin \varphi(x)$. Then, φ has a fixed point.*

PROOF. Suppose that φ does not have a fixed point. Then, since X is compact, we have points $x^1, \cdots, x^n \in X$ and their open neighbourhoods $U(x^1), \ldots, U(x^n)$ in X such that $\bigcup_{t=1}^n U(x^t) = X$, and for each $t = 1, \cdots, n$, there is a non-empty compact valued upper semi-continuous correspondence $p^{x^t} : U(x^t) \to E'$ satisfying that $\forall z \in U(x^t)$, $\forall q \in p^{x^t}(z)$, $\forall v \in \varphi(z) - z$, $\langle q, v \rangle > 0$. Let $\beta_t : X \to [0,1], t = 1, \cdots, n$, be a partition of unity subordinated to $\{U(x^1), \ldots, U(x^n)\}$. For each $z \in X$, define a set $q(z) \subset E'$ as $q(z) = \sum_{t=1}^n \beta_t(z) p^{x^t}(z)$. The set $q(z)$ is non-empty and compact, and the correspondence $q : X \to E'$ is upper semi-continuous. Let $\Phi(z) = \{y \in X | \langle q, y - z \rangle > 0 \text{ for all } q \in q(z)\}$ for each $z \in X$. Then, for all $y \in \varphi(z)$, and for all t such that $z \in U(x^t)$, we have

[1] The mathematical results in this paper is an extension of theorems in Urai (2000) and Urai-Hayashi (2000). See also these papers for other game theoretic applications and market equilibrium existence results.

$\langle q^t, y - z\rangle > 0$ for all $q^t \in p^{x^t}(z)$, so that $\sum_{t=1}^n \beta_t(z)\langle q^t, y - z\rangle = \langle \sum_{t=1}^n \beta_t(z) q^t, y - z\rangle > 0$ for all $q^t \in p^{x^t}(z)$ for all t. That is, for all $y \in \varphi(z)$, we have $\langle q, y - z\rangle > 0$ for all $q \in q(z)$. Hence, $\Phi : X \to X$ is a non-empty convex valued correspondence having no fixed point. On the other hand, for each $x \in X$ and for an arbitrary element $y^x \in \varphi(x)$, there is an open neighbourhood $V(x)$ of x in X satisfying $\forall z \in V(x)$, $y^x \in \Phi(z)$. (Indeed, since $y^x \in \Phi(x)$, we have $\langle q, y^x - x\rangle > 0$ for all $q \in q(x)$. Since $q(x)$ is compact, we have neighbourhood V of x and U of $q(x)$ such that for all $z \in V$ and $q \in U$, $\langle q, y^x - z\rangle > 0$ by the definition of the topology of compact convergence. Moreover, since $q(x)$ is upper semi-continuous, we may obtain a neighbourhood $V(x) \subset V$ of x such that $\forall z \in V(x)$, $q(z) \subset U$.) Hence, Φ has a fixed point by Browder's fixed point theorem (Browder (1968)), a contradiction. ∎

Note that the above theorem is a generalization of well known Kakutani-Fan-Glicksberg's fixed point theorem[2] since if $\varphi : X \to X$ is a non-empty compact convex valued upper semi-continuous correspondence, φ satisfies all conditions in Theorem 1. Indeed, if $\varphi(x)$ is compact and convex and if $x \notin \varphi(x)$, there is a $p \in E'$ such that $\langle p, z - x\rangle > 0$ for all $z \in \varphi(x)$ by the separation theorem. Then, the upper semi-continuity of φ means that φ has a fixed direction p near x.

In view of economics, the better set correspondence for a continuously differentiable quasi-concave utility function may be considered as a typical mapping having a continuous direction near every point. It is also easy to check that a better set correspondence which may be represented (at least locally) by quasi-concave utility functions have a compact valued upper semi-continuous direction near every point. Since the budget correspondence is ordinarily assumed to be non-empty compact convex valued upper semi-continuous, the generalized fixed point theorem enable us to treat these two mappings (the better set correspondence and the budget constraint correspondence) in a unified manner so that we may characterize the existence of economic equilibrium merely through conditions on local directions for these two mappings (see Theorem 2).

2.2 Existence of Equilibrium for an Abstract Economy

Let us define an *abstract economy* as a list, $(X_i, P_i, K_i)_{i \in I}$, where I is the finite index set of *agents*, X_i is the *choice set* of i for each $i \in I$, $K_i : X = \prod_{j \in I} X_j \to X_i$ is the *constraint correspondence* of i, and $P_i : X = \prod_{j \in I} X_j \to X_i$ is the *preference correspondence* of i. For each $i \in I$ and $x \in X$, we say that x_i is a *maximal point for P_i under K_i* iff $x_i \in K_i(x)$

[2] Let X be a non-empty compact subset of a locally convex Hausdorff topological vector space. Every non-empty closed convex valued upper semi-continuous correspondence $\varphi : X \to X$ has a fixed point $x^* \in \varphi(x^*)$. (c.f. Glicksberg (1952).)

and $P_i(x) \cap K_i(x) = \emptyset$.[3] An *equilibrium* of the abstract economy $(X_i, P_i, K_i)_{i \in I}$ is a point $\bar{x} \in X = \prod_{i \in I} X_i$ such that for each $i \in I$, $\bar{x}_i \in K_i(\bar{x})$ and $P_i(\bar{x}) \cap K_i(\bar{x}) = \emptyset$, i.e., a maximal point for P_i under K_i for all $i \in I$.

The second result of this paper is the following equilibrium existence theorem. Assume that for each $i \in I$, X_i is a subset of locally convex Hausdorff topological vector space E having topological dual E'. We say that a correspondence $\varphi_i : X \to X_i$ has a *compact valued upper semi-continuous direction near* $x \in X$ if there exist a neighbourhood $U(x)$ of $x \in X$ and a non-empty compact valued upper semi-continuous correspondence $p_i^x : U(x) \to E'$ such that $\langle q, y_i - z_i \rangle > 0$ for all $z \in U(x)$, $q \in p_i^x(z)$, and $y_i \in \varphi_i(z)$.[4] The next theorem characterize the existence of equilibrium merely through conditions on local directions of correspondences, P_i's and K_i's. The condition for P_i includes all cases such that preferences are locally represented by continuous utility functions. The condition for K_i includes all cases with continuous closed convex valued upper semicontinuous constraint correspondences.

THEOREM 2. *Suppose that abstract economy $(X_i, P_i, K_i)_{i \in I}$ satisfies the following conditions:*

(A1) *For each $i \in I$, X_i is a non-empty compact convex subset of locally convex Hausdorff topological vector space E having a topological dual E'.*

(A2) *For each $i \in I$, $P_i : X \to X_i$ is a (possibly empty valued) correspondence having a compact valued upper semi-continuous direction p_i^x near every $x \in X$.*

(A3) *For each $i \in I$, $K_i : X \to X_i$ is a non-empty valued correspondence having a compact valued upper semi-continuous direction q_i^x near every $x \in X$ such that $x_i \notin K_i(x)$.*

(A4) *For each $i \in I$, the set $\{x \in X | P_i(x) \cap K_i(x) \neq \emptyset\}$ is open in X.*[5]

Then, $(X_i, P_i, K_i)_{i \in I}$ has an equilibrium.

The theorem says that as long as the feasible set and the better set at a certain action level have continuous directions, (i) the convexity of these sets and the global continuity for

[3] In the following, given a vector $x \in X = \prod_{i \in I} X_i$, we denote by x_i the i-th coordinate of x as long as there is no ambiguity.

[4] The definition in the previous section may be identified with the special case such that $I = \{i\}$ and $X = X_i$.

[5] Under (A2) and (A3), condition (A4) may be dropped if "for each $x \in X$, $K_i(x)$ is a subset of the closure of the interior of K_i" and "for each $x \in X$, the value, $K_i(x)$, is not equal to X_i and does not depend on the i-th component, x_i, of x." In view of economics, these settings are quite natural. Of course, (A4) is automatically satisfied if K_i is lower semicontinuous and P_i has an open graph.

these mappings are not necessary, and (ii) merely the continuous local direction property of each mapping completely characterize the existence of equilibrium.

We provide a proof of the theorem under more general settings in the next corollary. In corollary 2.1, if we take $M_i(x)$ as $M_i(x) = K_i(x)$, condition (B5) is nothing but a tautology, hence, we have Theorem 2. The corollary is used in the next section, where the existence of the default leads to more serious non-convexity for K_i.

COROLLARY 2.1. *Suppose that abstract economy $(X_i, P_i, K_i)_{i\in I}$ satisfies the following conditions:*

(B1) *For each $i \in I$, X_i is a non-empty compact convex subset of locally convex Hausdorff topological vector space E having a topological dual E'.*

(B2) *For each $i \in I$, $P_i : X \to X_i$ is a (possibly empty valued) correspondence having a compact valued upper semi-continuous direction p_i^x near every $x \in X$.*

(B3) *For each $i \in I$, $K_i : X \to X_i$ is a correspondence such that there is a non-empty valued correspondence $M_i : X \to X_i$, $M_i(z) \subset K_i(z)$ for all $z \in X$, having a compact valued upper semi-continuous direction q_i^x near every $x \in X$ such that $x_i \notin K_i(x)$.*

(B4) *For each $i \in I$, the set $\{x \in X | P_i(x) \cap K_i(x) \neq \emptyset\}$ is open in X.*

(B5) *For each $x \in X$ and $i \in I$ such that $x_i \notin K_i(x)$ and $P_i(x) \cap K_i(x) = \emptyset$, there exists an open neighbourhood $U(x)$ of x such that for all $z \in U(x)$, $P_i(z) \cap K_i(z) \neq \emptyset$ implies $P_i(z) \cap M_i(z) \neq \emptyset$.*

Then, $(X_i, P_i, K_i)_{i\in I}$ has an equilibrium.

PROOF. In this proof, we denote by $V(z_i, Q)$ the set $\{y_i \in X_i | \langle q, y_i - z_i \rangle > 0 \text{ for all } q \in Q\}$, the set of direction Q at z_i for each $z_i \in X_i \subset E$ and $Q \subset E'$. Assume that there is no equilibrium for (X_i, P_i, K_i). Then, for each $x = (x_i)_{i\in I} \in X$, there is at least one $i \in I$ satisfying one and only one of the following three conditions:

(a) $K_i(x) \cap P_i(x) \neq \emptyset$.

(b) x is an iterior point of $\{z \in X | K_i(z) \cap P_i(z) = \emptyset \text{ and } z_i \notin K_i(z)\}$.

(c) x is a boundary point of $\{z \in X | K_i(z) \cap P_i(z) = \emptyset \text{ and } z_i \notin K_i(z)\}$.

In the following, we fix such an agent $i(x) \in I$ (satisfying one and only one of the conditions, (a), (b), and (c)) for each $x \in X$. For each x and case (a)–(c) of $i(x)$, define, locally, on a certain open neighbourhood $U(x)$ of x, a correspondence $\varphi^x : U(x) \to X_{i(x)}$ as

$$\begin{aligned} \varphi^x(z) &= V(z_{i(x)}, p^x_{i(x)}(z)) \text{ for case (a)}, \\ \varphi^x(z) &= V(z_{i(x)}, q^x_{i(x)}(z)) \text{ for case (b), and} \\ \varphi^x(z) &= V(z_{i(x)}, q^x_{i(x)}(z)) \text{ for case (c)}, \end{aligned}$$

where $p^x_{i(x)}$ for case (a) is the local direction for P_i on a certain open set $U(x)$ such that $z \in U(x)$ implies $K_i(z) \cap P_i(z) \neq \emptyset$ (use (B4)), $q^x_{i(x)}$ for case (b) is the local direction for M_i on a certain open set $U(x)$ such that $z \in U(x)$ implies $K_i(z) \cap P_i(z) \neq \emptyset$ and $z_i \notin K_i(z)$, and $q^x_{i(x)}$ for case (c) is the local direction for M_i on a certain open neighbourhood $U(x)$ of x. Under (B2) and (B4), we may chose $U(x)$ so that $\varphi^x(z) \supset K_{i(x)}(z) \cap P_{i(x)}(z) \neq \emptyset$ for all $z \in U(x)$ for case (a). Moreover, under (B3), we may chose $U(x)$ so that $\varphi^x(z) \supset M_{i(x)}(z) \neq \emptyset$ for all $z \in U(x)$ for cases (b) and (c). Note also that for each $x \in X$, $\varphi^x : U(x) \to X_{i(x)}$ is convex valued and $z_{i(x)} \notin \varphi^x(z)$ for each $z \in U(x)$. Since X is compact, there is a finite subcovering of the covering $\{U(x)\}_{x \in X}$ of X. Let $U(x^1), \ldots, U(x^n)$ be such a subcovering, and let $i^1, \ldots, i^n$ be names of agents fixed respectively for $x^1, \ldots, x^n$ in defining $U(x^1), \ldots, U(x^n)$ and $\varphi^{x^1}, \ldots, \varphi^{x^n}$. Moreover, for each $t = 1, \ldots, n$, let correspondence $p^t : U(x^t) \to E'$ be $p^{x^t}_{i(x^t)}$ for case (a) and be $q^{x^t}_{i(x^t)}$ for cases (b) and (c). Let $\beta_t : X \to [0,1], t = 1, \ldots, n$, be a partition of unity subordinated to $U(x^1), \ldots, U(x^n)$, satisfying that $(\beta_t(x) > 0) \iff (x \in U(x^t))$. For each x and i, denote by T_i and S_x the set of indices $\{t | i = i^t\}$ and $\{t | x \in U(x^t)\}$, respectively, and define a correspondence $\Phi : X \to X$ as $\Phi(x) = \prod_{i \in I} \Phi_i(x)$, where $\Phi_i(x) = V_i(x_i, \sum_{t \in S_x \cap T_i} \frac{\beta_t(x)}{\sum_{s \in S_x \cap T_i} \beta_s(x)} p^t(x))$ if $S_x \cap T_i \neq \emptyset$, and $\Phi_i(x) = X_i$ if $S_x \cap T_i = \emptyset$. Note that since for each x there is at least one i such that $x \in U(x^t)$ and $i = i^t$, (i.e., $S_x \cap T_i \neq \emptyset$), the mapping Φ has no fixed point. On the other hand, the mapping Φ is clearly convex valued. Moreover, Φ is non-empty valued since every φ^{x^t} is non-empty valued on $U(x^t)$ and since $\Phi_i(x) \supset \bigcap_{t \in S_x \cap T_i} V_i(x_i, p^t(x))$ for each i and x such that $S_x \cap T_i \neq \emptyset$. (Indeed, $\bigcap_{t \in S_x \cap T_i} V_i(x_i, p^t(x))$ is clearly non-empty when all elements of $T_i \cap S_x$ are type (b) or (c). We may also check the set to be non-empty when the elements of $T_i \cap S_x$ are type (a) or (c), under condition (B5). Note that $U(x^t)$'s of type (a) and type (b) never intersect.) Furthermore, Φ has a compact valued upper semi-continuous direction near each $x \in X$ since $S_x \cap T_i \neq \emptyset$ implies that $\Phi_i(z) \subset V_i(z_i, \sum_{t \in S_x \cap T_i} \frac{\beta_t(z)}{\sum_{t \in S_x \cap T_i} \beta_s(z)} p^t(z))$ for all $z \in \bigcap_{t \in S_x \cap T_i} U(x^t)$. Hence, Φ has a fixed point by Theorem 1, so that we have a contradiction. ∎

3 APPLICATION FOR THE DEFAULT ECONOMY

In the previous section, we have shown the theorem on the existence of equilibria for an abstract economy under non-convex constraint correspondence. We apply in this section the theorem to prove the existence of equilibria allowing for default and/or bankruptcy.

Under general equilibrium settings, problems with asymmetric information (including default and/or bankruptcy as well as adverse selection and moral hazard) are closely related to non-convex agents' budget constraints since we may settle the problem into the situation that the constraint (the price) as a buyer is different from the constraint from the view point of a seller. Especially, the non-convexity problem in economies with default and/or bankruptcy is classical and has been treated in many literature (see, e.g., Green (1974), Grandmont (1982), Eichberger (1989)).

Recent several authors (see, e.g., Zame (1993), Dubey-Geanakoplos-Shubik (2000)) have avoided the problem by considering a completely anonymous asset market in which the average default rate (for each asset in the market) is perfectly expected and each agent is allowed simultaneously to buy and sell (go long and short in) the same asset. It is interesting that they have changed the problem about moral hazard into the choice problem under the taste for morals of each agent. Such an approach, however, (1) fails to describe an accidental or unintended default and/or bankruptcy, (2) implicitly changes the concept of standard economic equilibrium where for each commodity, demand and supply are treated in a total net amount, and, especially, (3) they have used the concept of the "complete anonymity" as a "necessary" condition for the existence of equilibria. Of course, it is more desirable to show an existence of equilibrium directly with non-convex budget constraints with or without the complete anonymity and perfect foresights.

To overcome the non-convexity problem on budget constraints in economies with default or bankruptcy, many authors use the smoothing method under continuum of traders or the methods of Starr (1969). An exceptional approach (in temporary general equilibrium model) with money and the central bank under a special bankruptcy rule is Eichberger (1989).[6] Also, Green (1974) proved the existence of an approximate equilibrium, following the Starr method, by introducing a bankruptcy rule and disutility from the extent of bankruptcy. In the following, we treat the problem by using the concept we have used in sections 1 and 2, the direction of correspondences.

3.1 The Default Economy

Now let us consider a two-period economy, period 1 (present) and 2 (future). There are m types of agents indexed by $i = 1, \dots, m$. In periods 1 and 2 there are, respectively, ℓ types of real commodities. We also assume that there are K types of assets indexed by $k = 1, \dots, K$ that is sold and purchased in the asset market in each period.[7] Asset k, $A^k \in R^\ell$, specifies bundles of goods to be delivered in period 2. We assume that the payoff of each asset is specified to a certain commodity. Hence, in each period, two types

[6] A further development of the model in Eichberger (1989) is treated in Yoshimachi (1999).

[7] We suppose that the number of states in period 2 is one for the sake of notational simplicity.

of markets, the *spot market* for ℓ types of real commodities and the *asset market* for K types of assets, exist.

Prices for commodities in the spot market and the asset market in period 1 are denoted by $p^1 = (p^{11}, \dots, p^{1\ell}) \in R^{\ell}_{+}$ and $q^1 = (q^{11}, \dots, q^{1K}) \in R^K_{+}$, respectively. The *set of prices* in period 1 is defined by $\Delta = \{(p^1, q^1) \in R^{\ell}_{+} \times R^K_{+} : p^{11} + \cdots + p^{1\ell} + q^{11} + \cdots + q^{1K} = 1\}$ be the set of prices.

Agent i has an *initial endowment,* $\omega_i = (\omega_i^1, \omega_i^2) \in R^{\ell} \times R^{\ell}$, of real commodities, and a *consumption set,* $X_i \subset R^{\ell} \times R^{\ell}$. We assume that X_i is a closed convex subset bounded from below such that $X_i \subset X_i + R^{2\ell}_{+}$ for each i. The preference of agent i which may possibly be non-ordered is represented at each $x \in \prod_{i=1}^m X_i$ by a continuous concave utility function $u_i^x : X_i \to R$. We also assume that at each $x \in X$, the value of utility function u_i^x at $y_i \in X_i$, i.e., $u_i^x(y_i)$ is continuous with respect to $(x, y_i) \in X \times X_i$. In order to determine his plan, agent i forecast the prices of commodities and the default rate of each asset in period 2. We assume that agent i has continuous *expectation functions* for prices,

$$p_i : \quad \Delta \times \Delta \times [0,1]^K \to \Delta \times \Delta \times [0,1]^K, \tag{1}$$

$$q_i : \quad \Delta \times \Delta \times [0,1]^K \to \Delta \times \Delta \times [0,1]^K, \tag{2}$$

and for *delivery rates*,

$$\rho_i : \Delta \times \Delta \times [0,1]^K \to \Delta \times \Delta \times [0,1]^K.^8 \tag{3}$$

For a list of other person's consumption plans, x_j, $j = 1, \dots, m$, $j \neq i$, agent i chooses $x_i = (x_i^1, x_i^2) \in X_i$, $\theta_i = (\theta_i^1, \theta_i^2) \in R^K_{+} \times R^K_{+}$, and $\varphi_i = (\varphi_i^1, \varphi_i^2) \in R^K_{+} \times R^K_{+}$ so that x_i maximize his utility function $u_i^{(x_1,\dots,x_i,\dots,x_m)}$ under his budget constraints, where θ_i and φ_i indicate the quantity of assets *purchased* and *sold*, respectively. Notice that for each agent i, each asset k has a *limit on the quantity for sales*, Q_i^{tk}, $\varphi_i^{tk} \leq Q_i^{tk}$, $t = 1, 2$. We now define the *budget set* of agent i, $K_i(p, q, \rho)$. (An element $(p, q, \rho) = ((p^1, p^2), (q^1, q^2), \rho)$ of $\Delta \times \Delta \times [0,1]^K$ will be denoted by s in the following.) In period 1, agent i should satisfy

$$p^1 \cdot x_i^1 + q^1 \cdot \theta_i^1 \leq p^1 \cdot \omega_i^1 + q^1 \cdot \varphi_i^1. \tag{4}$$

In the same way, we may suppose an expected budget constraint at period 2 for i to be

$$\begin{aligned} & p_i^2(s) \cdot (x_i^2)^+ + q_i^2(s) \cdot \theta_i^2 \\ & \quad \leq \max \left\{ 0, p_i^2(s) \cdot (\omega_i^2 + (x_i^2)^-) + q_i^2(s) \cdot \varphi_i^2 + \sum_{k=1}^K p_i^2(s) A^{1k} (\rho_i^k(s) \theta_i^{1k} - \varphi_i^{1k}) \right\}, \end{aligned} \tag{5}$$

[8] In the domains and ranges of these functions, the first Δ denotes the set of prices in period 1, the second Δ denotes the set of prices in period 2, and $[0,1]^K$ denotes the set of delivery rates. Of course, the case with rational expectations is treated as the special (identity function) case of these expectation functions. For the concept of delivery rates, see Zame (1993) and Dubey-Geanakoplos-Shubik (2000).

where $(x_i^2)^+ = \sup\{0, x_i^2\}$, $(x_i^2)^- = \sup\{0, -x_i^2\}$, and the max operation allows for the possibility of default. Also, we have to allow for the bound on sales for asset k for agent i and a natural condition on the quantity of assets purchased and sold, respectively.

$$\forall t, \forall k, \qquad \varphi_i^{tk} \leq Q_i^{tk}, \tag{6}$$

$$\forall t, \forall k, \qquad \theta_i^{tk} \cdot \varphi_i^{tk} = 0. \tag{7}$$

Thus, the budget set of agent i is

$$K_i(p, q, \rho) = \{(x_i, \theta_i, \varphi_i) | (x_i, \theta_i, \varphi_i) \text{ satisfies } (4), (5), (6) \text{ and } (7) \text{ under } (p, q, \rho)\}. \tag{8}$$

As in Dubey, Geanakoplos, and Shubik (2000), we define for each agent i a constant, $\lambda_i \in R_+$, the *real default penalty on i*. We assume that the default penalties are assessed directly in terms of utility of agent i and are proportional to the size of the default. Hence, under $s = (p, q, \rho)$, the utility level of i for $(y_i, \theta_i, \varphi_i)$ at $x \in X$, $W_i^x(y_i, \theta_i, \varphi_i, s)$ is as follows:

$$\begin{aligned} W_i^x(y_i, \theta_i, \varphi_i, s) &= u_i^x(y_i) + U_i^x(\theta_i^2, \varphi_i^2) \\ &-\lambda_i \max\left\{0, \sum_{k=1}^{K} p_i^2(s) A^{1k}(\varphi_i^{1k} - \rho_i^k(s)\theta_i^{1k}) - p_i^2(s)\cdot(\omega_i^2 + (x_i^2)^-) - q_i^2(s)\cdot\varphi_i^2\right\}, \end{aligned} \tag{9}$$

where constraints (4)–(7) are not taken into consideration and the term $U_i^x(\theta_i^2, \varphi_i^2)$ denotes an appropriately defined *indirect utility* for assets in the second period. In the following, we neglect every terms on θ_i^2 and φ_i^2 in equation (9) for the sake of simplicity. When condition (4) is satisfied, the second entry in the last term in (9) (i.e., $\sum_{k=1}^K p_i^2(s)A^{1k}(\varphi_i^{1k} - \rho_i^k(s)\theta_i^{1k}) - p_i^2(s)\cdot(\omega_i^2 + (x_i^2)^-) - q_i^2(s)\cdot\varphi_i^2$,) represents the size of the default of i associated with the action and plan $(x_i, \theta_i, \varphi_i)$ under s. We denote the amount by $D_i(x_i, \theta_i, \varphi_i, s)$.

Denote by $\mathcal{E} = (X_i, \omega_i, u_i, (A^k)_{k=1}^K, \lambda_i, (Q_i^k)_{k=1}^K)_{i=1}^m$ the economy stated in the above. An *equilibrium* for economy $\mathcal{E}$ is $(x, \theta, \varphi, p^1, q^1, p^2, q^2, \rho) \in \prod_{i=1}^m X_i \times (R_+^K)^2 \times (R_+^K)^2 \times \Delta \times \Delta \times [0,1]^K$ such that for $i \in \{1, \ldots, m\}$, $(x_i, \theta_i, \varphi_i) \in \arg\max W_i^x(y_i, \theta_i, \varphi_i)$ over i's budget under (p, q, ρ), and *market clearing conditions* together with a *specification of expectation functions* such that

$$\sum_{i=1}^m (x_i^1 - \omega_i^1) = 0 \text{ and } \sum_{i=1}^m (\theta_i^1 - \varphi_i^1) = 0, \tag{10}$$

$$\forall i = 1, \ldots, m, \ p_i^1 \text{ and } q_i^1 \text{ are identity functions}, \tag{11}$$

(for the *short run equilibrium*), or

$$\textstyle\sum_{i=1}^m (x_i - \omega_i) = 0 \text{ and } \sum_{i=1}^m (\theta_i - \varphi_i) = 0, \tag{12}$$

$$\forall k = 1, \ldots, K, \ 1 - \rho^k = (\sum_{i=1}^m D_i(x_i, \theta_i, \varphi_i, s))/(\sum_{i=1}^m p_i^2(s) A^{1k}\varphi_i^{1k}), \tag{13}$$

$$\forall i = 1, \ldots, m, \ p_i, q_i, \rho_i \text{ are identity functions}, \tag{14}$$

(for the *long run equilibrium* with perfect foresights). As in Eichberger (1989) and Dubey, Geanakoplos, and Shubik (2000), consumers or agents having budget constraints like the above $K_i(p, q, \rho)$ have a common problem that the budget set may not be convex as long as we suppose the natural setting (4).

3.2 Existence of Equilibrium

For each $\epsilon > 0$ sufficiently small, we define an abstract economy $\mathscr{E}_\epsilon$ for the economy $\mathcal{E}$ as follows.

Let Δ_ϵ be the set of all vectors in Δ whose all coordinates are greater than or equal to $\epsilon > 0$, and denote by S the set $\Delta \times \Delta \times [0,1]^K$. If we restrict the set of prices for economy $\mathcal{E}$ on the compact set $S_\epsilon = \Delta_\epsilon \times \Delta_\epsilon \times [\epsilon, 1]^K$, then the set of all consumption and asset holding levels that are not associated with default is also confined in a certain compact cube, $C_\epsilon \subset (R^\ell \times R^K_+) \times (R^\ell \times R^K_+)$. Under the default penalty, we may take such a C_ϵ so large that for each $x \in \prod_{i=1}^m X_i$, every action $(y_i, \theta_i, \varphi_i)$ which maximizes W_i^x in C_ϵ maximizes W_i^x in $(X_i \times R^K_+ \times R^K_+) \times (X_i \times R^K_+ \times R^K_+)$.[9] For $i = 1, \dots, m$, let $Z_i = (X_i \times R^K_+ \times R^K_+) \times (X_i \times R^K_+ \times R^K_+)$, and let the choice set $Z_{i\epsilon}$ of abstract economy $\mathscr{E}_\epsilon$ as $C_\epsilon \cap Z_i$.

For each agent $i = 1, \dots, m$, define the preference correspondence $P_{i\epsilon}$ as the better set correspondence at $x \in \prod_{i=1}^m X_i$ under W_i^x on $Z_{i\epsilon}$, and define the constraint correspondence $K_{i\epsilon}$ as (8) on $Z_{i\epsilon}$.

For 0-th agent, an auctioneer, define a utility (payoff) function, $W_0(x, \theta, \varphi, p, q, \rho)$, as

$$W_0(x, \theta, \varphi, p, q, \rho) = p^1 \sum_{i=1}^m (x_i^1 - \omega_i^1) + q^1 \sum_{i=1}^m (\theta_i^1 - \varphi_i^1), \tag{15}$$

for the short run equilibrium, and as

$$W_0(x, \theta, \varphi, p, q, \rho) = p\textstyle\sum_{i=1}^m (x_i - \omega_i) + q \displaystyle\sum_{i=1}^m (\theta_i - \varphi_i), \tag{16}$$

for the long run equilibrium with perfect foresights. For agent 0, define the choice set as $Z_{0\epsilon} = S_\epsilon$, $(Z_0 = S)$, and the constraint correspondence as $K_{0\epsilon}(x, \theta, \varphi, p, q, \rho) = Z_{0\epsilon}$.

Let the set of players for $\mathscr{E}_\epsilon$ be $I = \{0, 1, \dots, m\}$. Then, the following two additional structures (the minimum wealth structure and the minimum delivery rate structure) enable us to show that for a sufficiently small $\epsilon > 0$, an equilibrium for $\mathscr{E}_\epsilon$ is an equilibrium for the default economy $\mathcal{E}$.

[9] As stated before, the last two terms $R^K_+ \times R^K_+$ (the amount of assets purchased and sold in the second period) together with their all entries in W_i^x will be neglected in the following for the sake of simplicity.

Assumption (Minimum Wealth) : *There is a continuous wealth transfer mechanism (a function $\tau^w : \Delta \times \Delta \times [0,1]^K \to R^m$) which assures for each agent i and (p,q,ρ), a wealth level which is strictly greater than the minimum level under (p,q,ρ).*

Assumption (Minimum Delivery Rate) : *There is a continuous insurance mechanism (a function $\tau^d : \Delta \times \Delta \times [0,1]^K \to [0,1]^K$ financed by the transfer of wealth) which assures for each asset k and (p,q,ρ), a strictly positive minimum delivery rate under (p,q,ρ).*

It is true that in considering these two mechanisms, we have to rewrite the budget for i (equations (4) and (5)) and reconsider all related assertions. We shall omit, however, the process since (i) it is a routine task (e.g., use a continuous transfer function on endowments $\tau : \Delta \times \Delta \times [0,1]^K \to R^\ell \times R^\ell$), and (ii) they are not indispensable structures.[10]

We now state the main theorem in this section. Let $M_i(s)$ be the set of elements of $K_i(s)$ such that $D(x_i, \theta_i, \varphi_i, s) = 0$ for each $s = (p,q,\rho)$. It is clear that $M_i(s)$, the set of *moral actions and plans*, is convex. It is also easy to verify that for each $i = 1, \ldots, m$, $W_i^x(y_i, \theta_i, \varphi_i, s)$ is continuous with respect to x, y_i, θ_i, φ_i, and s, (see equation (9) for the definition of W_i^x,) and at each x and s, the better set correspondence defined by W_i^x on $Z_\epsilon = \prod_{i=0}^m Z_{i\epsilon}$ into $Z_{i\epsilon}$ is convex valued. Hence, each $P_{i\epsilon}$, $i = 1, \ldots, m$, is a convex valued correspondence having continuous local utility representation, so that has a compact valued upper semi-continuous direction at every point.

Theorem 3. *Assume the minimum wealth condition and the minimum delivery rate condition. Then, the economy $\mathcal{E} = (X_i, \omega_i, u_i, (A^k)_{k=1}^K, \lambda_i, (Q_i^k)_{k=1}^K)_{i=1}^m$ has a long run (perfect foresight) equilibrium if the following condition is satisfied.*

> (Local Directions for Morals) *At each $(x_i, \theta_i, \varphi_i)$, $i = 1, \ldots, m$, and $s = (p,q,\rho)$ such that $(x_i, \theta_i, \varphi_i) \notin K_i(s)$ and $P_{i\epsilon}(x,\theta,\varphi) \cap K_i(s) = \emptyset$, there is an open neighbourhood $U(s,x,\theta,\varphi)$ of $(s,x,\theta,\varphi) \in Z = \prod_{i=0}^m Z_i$ such that for all $z \in U(s,x,\theta,\varphi)$, $P_{i\epsilon}(z_1, \ldots, z_m) \cap K_i(z_0) \neq \emptyset$ implies $P_{i\epsilon}(z_1, \ldots, z_m) \cap M_i(z_0) \neq \emptyset$.*

Adding to the above condition, if we suppose for each i and k, that the range of ρ_i^k is a subset of strictly positive reals, then a short run equilibrium exists.

Proof. The compactness of $\Delta \times \Delta \times [0,1]^K$ assures that there are positive numbers $\delta > 0$ and $\mu > 0$ such that all agents have an ability to pay (without default) greater than δ and all delivery rates are greater than μ. For the case with short run equilibrium, we may suppose without loss of generality that the range of ρ_i^k is a subset of $[\mu, 1]$. Since

[10] It is always possible to replace these two mechanisms with more individualistic conditions on endowments, weaker boundary conditions, and standard limit arguments. The mechanism on default penalties, however, is rather essential in this paper. The ordinary cubic truncation method fails to assure demands under truncation to be true maximal elements under the non-convexity of constraints.

utility functions are strictly positive and $X_i \supset X_i + R^{2\ell}_+$, and since feasible utility levels are bounded, the existence of δ and μ means that all equilibrium prices, if such exist, should be in Δ_ϵ for an $\epsilon > 0$ sufficiently small. Moreover, since the aggregate supply in economy $\mathcal{E}$ has an upper bound, and since the existence of an upper bound for the aggregate demand means the existence of an upper bound for the utility levels (without default) for all i, we may take such an ϵ so small that under any boundary price $(p, q) \in \Delta_\epsilon \times \Delta_\epsilon$ $((p^1, q^1) \in \Delta_\epsilon$, for the short run case) and a delivery rate $\rho^k \geq \mu$, the summation of excess demands of all commodities (in period 1, resp., for the short run case) under (p, q, ρ) is positive. (Indeed, by taking ϵ arbitrarily small, since C_ϵ includes all non default actions and plans, the utility level under boundary prices can be taken arbitrarily large. Hence, the total excess demand under such boundary prices cannot be bounded.) That is, the price system $\bar{p} = (\frac{1}{\ell}, \ldots, \frac{1}{\ell})$ and $\bar{q} = (\frac{1}{K}, \ldots, \frac{1}{K})$ ($\bar{p}^1$ and $\bar{q}^1$, resp., for the short run) appreciates all boundary excess demands positively. This fact together with the Walras' law in all periods (in period 1, resp., for the short run) implies that the maximal point for player 0 in abstract economy $\mathscr{E}_\epsilon$ for such an ϵ is the market clearing price. Moreover, since we have chosen the bound C_ϵ for $\mathscr{E}_\epsilon$ so that for each i, every solution to the utility maximization in $\mathscr{E}_\epsilon$ to be a solution in $\mathcal{E}$, we have shown that an equilibrium for $\mathscr{E}_\epsilon$ is an equilibrium for $\mathcal{E}$.

Next, we have to show that abstract economy $(Z_{i\epsilon}, P_{i\epsilon}, K_{i\epsilon})_{i\in I}$ for economy $\mathcal{E}$ satisfies all conditions in Corollary 2.1. Condition (B1) is clearly satisfied. For consumers, the minimum wealth condition assures the continuity of K_i and M_i together with the non-emptiness for the value of M_i. By the convexity and closedness for the value of M_i, (B3) is automatically satisfied. Moreover, as stated before, the better set correspondence under W^x_i has a compact valued upper semi-continuous direction near every point, so that (B2) is also satisfied. Furthermore, the continuity of $W^x_i(y_i, \theta_i, \varphi_i, s)$ with respect to x, y_i, θ_i, φ_i, and s means that $P_{i\epsilon}$ has an open graph. This, together with the continuity of K_i, assures (B4). It is clear that the condition on Local Directions for Morals assures condition (B5). For the auctioneer, the upper semicontinuity and closed convex non-empty valuedness of price transformation correspondence satisfies condition (B2). Moreover, by considering $M_i(s) = K_i(s) = Z_{i\epsilon}$ for all s, (B3), (B4), and (B5) are automatically satisfied. Hence, $\mathscr{E}_\epsilon$ satisfies all conditions in Theorem 2. ∎

A typical example for preferences assuring the condition of local directions in Theorem 3 is the family of CES utility functions on the positive orthant consumption sets. For any utility function, however, we may expect that an appropriate level of utility punishment always assures the condition since at each x such that $x_i \notin K_i(x)$, the better set at x_i eventually include a point in $M_i(x)$ for all λ_i sufficiently large.

(Graduate School of Economics, Osaka University)

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